

# STABILITY OF FROBENIUS DIRECT IMAGES OVER SURFACES

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**ABSTRACT.** Let  $X$  be a smooth projective surface over an algebraically closed field  $k$  of characteristic  $p > 0$  with  $\Omega_X^1$  semistable and  $\mu(\Omega_X^1) > 0$ . For any semistable (resp. stable) bundle  $W$  of rank  $r$ , we prove that  $F_*W$  is semistable (resp. stable) when  $p \geq r(r-1)^2 + 1$ .

## 1. INTRODUCTION

Let  $X$  be a smooth projective variety of dimension  $n$  over an algebraically closed field  $k$  with  $\text{char}(k) = p > 0$ . The absolute Frobenius morphism  $F_X : X \rightarrow X$  is induced by  $\mathcal{O}_X \rightarrow \mathcal{O}_X, f \mapsto f^p$ . Let  $F : X \rightarrow X_1 := X \times_k k$  denote the relative Frobenius morphism over  $k$ . This simple endomorphism of  $X$  is of fundamental importance in algebraic geometry over characteristic  $p > 0$ . One of the themes is to study its action on the geometric objects on  $X$ .

Recall that a torsion free sheaf  $\mathcal{E}$  is called semistable (resp. stable) if  $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$  (resp.  $\mu(\mathcal{E}') < \mu(\mathcal{E})$ ) for any nontrivial proper subsheaf, where  $\mu(\mathcal{E})$  is the slope of  $\mathcal{E}$  (see Definition 1 in Section 2). Semistable sheaves are basic constituents of torsion free sheaves in the sense that for any torsion free sheaf  $\mathcal{E}$  admits a unique filtration

$$\text{HN}_\bullet(\mathcal{E}) : 0 = \text{HN}_0(\mathcal{E}) \subset \text{HN}_1(\mathcal{E}) \subset \cdots \subset \text{HN}_k(\mathcal{E}) = \mathcal{E},$$

which is the so called Harder-Narasimhan filtration, such that

- (1)  $\text{gr}_i^{\text{HN}}(\mathcal{E}) := \text{HN}_i(\mathcal{E})/\text{HN}_{i-1}(\mathcal{E})$  ( $1 \leq i \leq k$ ) are semistable;
- (2)  $\mu(\text{gr}_1^{\text{HN}}(\mathcal{E})) > \mu(\text{gr}_2^{\text{HN}}(\mathcal{E})) > \cdots > \mu(\text{gr}_k^{\text{HN}}(\mathcal{E}))$ .

The rational number  $I(\mathcal{E}) := \mu(\text{gr}_1^{\text{HN}}(\mathcal{E})) - \mu(\text{gr}_k^{\text{HN}}(\mathcal{E}))$ , which measures how far is a torsion free sheaf from being semistable, is called the instability of  $\mathcal{E}$ . It is clear that  $\mathcal{E}$  is semistable if and only if  $I(\mathcal{E}) = 0$ .

It is well known that  $F_*$  preserves the stability of vector bundles on curves of genus  $g \geq 1$  (see [5], [6], [7]). For the high dimension case, it is proved by X. Sun that instability of  $F_*W$  is bounded by instability of  $W \otimes T^\ell(\Omega_X^1)$  ( $0 \leq \ell \leq n(p-1)$ ) for any vector bundle  $W$  (see [6], [7]), and an upper bound of the instability  $I(W \otimes T^\ell(\Omega_X^1))$  is given in [4]. Especially for a surface  $X$  with  $\Omega_X^1$  semistable and  $\mu(\Omega_X^1) > 0$ , the

stability of  $F_*\mathcal{L}$  for a line bundle  $\mathcal{L}$  is proved by X. Sun (see [6]). But it is unknown whether  $F_*$  preserves the stability of a high rank vector bundle over a smooth projective surface. In this note, we prove that  $F_*W$  is semistable (resp. stable) when  $W$  is semistable (resp. stable) with some restriction on the characteristic  $p$  as following:

**Theorem 1.** *Let  $X$  be a smooth projective surface over an algebraically closed field  $k$  of characteristic  $p$  with  $\Omega_X^1$  semistable and  $\mu(\Omega_X^1) > 0$ . Let  $W$  be a semistable (resp. stable) vector bundle of rank  $r$ , then  $F_*W$  is also semistable (resp. stable) if  $p \geq r(r-1)^2 + 1$ .*

Here, we sketch the proof. By [6], there exists a canonical filtration of  $F^*(F_*W)$ :

$$0 = V_{2(p-1)+1} \subset V_{2(p-1)} \subset \cdots \subset V_1 \subset V_0 = F^*(F_*W)$$

with  $V_\ell/V_{\ell+1} \cong W \otimes T^\ell(\Omega_X^1)$  for  $0 \leq \ell \leq 2(p-1)$ . Let  $\mathcal{E} \subset F_*W$  be a nontrivial subsheaf such that  $F_*W/\mathcal{E}$  is torsion free, then the above filtration induces the following filtration (we assume  $V_m \cap F^*\mathcal{E} \neq 0$  and  $V_{m+1} \cap F^*\mathcal{E} = 0$ )

$$0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}.$$

Let

$$\mathcal{F}_\ell := \frac{V_\ell \cap F^*\mathcal{E}}{V_{\ell+1} \cap F^*\mathcal{E}} \subset \frac{V_\ell}{V_{\ell+1}}, \quad r_\ell = \text{rk}(\mathcal{F}_\ell).$$

Then, taking  $n = 2$  in the formula (4.10) of [7], we have

$$\mu(\mathcal{E}) - \mu(F_*W) = \sum_{\ell=0}^m r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \text{rk}(\mathcal{E})} - \frac{\mu(\Omega_X^1)}{p \cdot \text{rk}(\mathcal{E})} \sum_{\ell=0}^m (p-1-\ell)r_\ell.$$

If  $r_{2(p-1)} = r_0$ , there exists a subsheaf  $W' \subset W$  of rank  $r_{2(p-1)}$  such that  $\mathcal{F}_\ell \supseteq W' \otimes T^\ell(\Omega_X^1)$  for  $0 \leq \ell \leq 2(p-1)$  by [7]. The local computations in the proof of Theorem 4.7 of [7] imply  $r_\ell = \text{rk}(W' \otimes T^\ell(\Omega_X^1))$  for  $0 \leq \ell \leq 2(p-1)$ . Then, by (4.22) of [7], we have

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \frac{r_{2(p-1)}(\text{rk}(F_*W) - \text{rk}(\mathcal{E}))}{p \cdot \text{rk}(\mathcal{E}) \cdot \text{rk}(W)} (\mu(W') - \mu(W/W'))$$

Otherwise, we have  $r_0 > r_{2(p-1)}$  and

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \sum_{\ell=0}^m r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \text{rk}(\mathcal{E})} - \frac{(p-1)\mu(\Omega_X^1)}{p \cdot \text{rk}(\mathcal{E})}$$

by (4.10), (4.11) and (4.12) of [7].

The main part of this note is to give a upper bound of

$$\sum_{\ell=0}^m r_{\ell}(\mu(\mathcal{F}_{\ell}) - \mu(\frac{V_{\ell}}{V_{\ell+1}})),$$

which depends only on  $r$  and  $\mu(\Omega_X^1)$ .

## 2. PRELIMINARIES

Let  $X$  be a smooth projective surface. Fixed an ample divisor  $H$ , for a torsion free sheaf  $\mathcal{E}$  on  $X$ , we define the slope of  $\mathcal{E}$  by :

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H}{\text{rk}(\mathcal{E})},$$

where  $c_1(\mathcal{E})$  is the first Chern class of  $\mathcal{E}$  and  $\text{rk}(\mathcal{E})$  is the rank of  $\mathcal{E}$ .

**Definition 1.** A torsion free sheaf  $\mathcal{E}$  on  $X$  is called semistable (resp. stable) if for any subsheaf  $0 \neq \mathcal{E}' \subset \mathcal{E}$  with  $\mathcal{E}/\mathcal{E}'$  torsion free, we have

$$\mu(\mathcal{E}') \leq \mu(\mathcal{E}) \quad (\text{resp. } \mu(\mathcal{E}') < \mu(\mathcal{E})).$$

Let  $F : X \rightarrow X_1$  be the relative  $k$ -linear Frobenius morphism, where  $X_1 := X \times_k k$  is the base change of  $X/k$  under the Frobenius  $\text{Spec}(k) \rightarrow \text{Spec}(k)$ . Let  $W$  be a vector bundle on  $X$  and  $V = F^*(F_*W)$ .

**Definition 2.** Let  $V_0 := V = F^*(F_*W)$ ,  $V_1 = \ker(F^*(F_*W) \rightarrow W)$

$$V_{\ell+1} := \ker(V_{\ell} \xrightarrow{\nabla} V \otimes_{\mathcal{O}_X} \Omega_X^1 \rightarrow (V/V_{\ell}) \otimes_{\mathcal{O}_X} \Omega_X^1)$$

where  $\nabla : V \rightarrow V \otimes_{\mathcal{O}_X} \Omega_X^1$  is the canonical connection (see [1, Theorem 5.1]).

The above filtration has been fully studied in [6, Section 3], and the following theorem is a special case of [6, Theorem 3.7, Corollary 3.8] for surfaces.

**Theorem 2.** [6, Theorem 3.7, Corollary 3.8] Let  $X$  be a smooth projective surface over  $k$ , then the filtration defined above is

$$0 = V_{2(p-1)+1} \subset V_{2(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W) \quad (1)$$

which has the following properties

- (i)  $\nabla(V_{\ell}) \subset V_{\ell-1} \otimes \Omega_X^1$  for  $\ell \geq 1$ , and  $V_0/V_1 \cong W$ .
- (ii)  $V_{\ell}/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_{\ell}) \otimes \Omega_X^1$  are injective morphisms of vector bundles for  $1 \leq \ell \leq 2(p-1)$ , which induced isomorphisms  $V_{\ell}/V_{\ell+1} = W \otimes T^{\ell}(\Omega_X^1)$  where

$$T^{\ell}(\Omega_X^1) = \begin{cases} \text{Sym}^{\ell}(\Omega_X^1) & \text{when } \ell < p \\ \text{Sym}^{2(p-1)-\ell}(\Omega_X^1) \otimes \omega_X^{\ell-(p-1)} & \text{when } \ell \geq p. \end{cases}$$

Let  $\mathcal{E} \subset F_*W$  be a nontrivial subsheaf such that  $F_*W/\mathcal{E}$  is torsion free, then the canonical filtration (1) induces the filtration (we assume  $V_m \cap F^*\mathcal{E} \neq 0$  and  $V_{m+1} \cap F^*\mathcal{E} = 0$ )

$$0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}. \quad (2)$$

Let

$$\mathcal{F}_\ell := \frac{V_\ell \cap F^*\mathcal{E}}{V_{\ell+1} \cap F^*\mathcal{E}} \subset \frac{V_\ell}{V_{\ell+1}}, \quad r_\ell = \text{rk}(\mathcal{F}_\ell).$$

Then  $\mu(F^*\mathcal{E}) = \frac{1}{\text{rk}(\mathcal{E})} \sum_{\ell=0}^m r_\ell \cdot \mu(\mathcal{F}_\ell)$  and

$$\mu(\mathcal{E}) - \mu(F_*W) = \frac{1}{p \cdot \text{rk}(\mathcal{E})} \sum_{\ell=0}^m r_\ell (\mu(\mathcal{F}_\ell) - \mu(F^*F_*W)). \quad (3)$$

**Lemma 1.** ([7, Lemma 4.5]) *With the same notations in Theorem 2, we have*

$$\begin{aligned} \mu(F^*F_*W) &= p \cdot \mu(F_*W) = \frac{p-1}{2} K_X \cdot H + \mu(W), \\ \mu(V_\ell/V_{\ell+1}) &= \mu(W \otimes T^\ell(\Omega_X^1)) = \frac{\ell}{2} K_X \cdot H + \mu(W). \end{aligned}$$

By using the above lemma, we have

**Lemma 2.** ([6, Lemma 4.4]) *Keep the above notations. Then we have*

$$\mu(\mathcal{E}) - \mu(F_*W) = \sum_{\ell=0}^m r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \text{rk}(\mathcal{E})} - \frac{\mu(\Omega_X^1)}{p \cdot \text{rk}(\mathcal{E})} \sum_{\ell=0}^m (p-1-\ell)r_\ell. \quad (4)$$

The numbers  $r_\ell$  ( $0 \leq \ell \leq m$ ) are related by the following fact that  $V_\ell/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_\ell) \otimes \Omega_X^1$  induces injective morphisms

$$\mathcal{F}_\ell \xrightarrow{\nabla} \mathcal{F}_{\ell-1} \otimes \Omega_X^1 \quad (1 \leq \ell \leq m).$$

Using this fact, it is proved in [6] that

$$r_{2(p-1)-\ell} - r_\ell \geq 0 \quad (\ell \geq p-1).$$

Especially for  $\ell = 2(p-1)$ , we have  $r_0 \geq r_{2(p-1)}$ . The following lemma is implicitly in [7, Lemma 4.6].

**Lemma 3.** *If  $r_0 > r_{2(p-1)}$ , then we have*

$$\sum_{\ell=0}^m (p-1-\ell)r_\ell \geq (p-1).$$

*Proof.* When  $m \leq p-1$ , it is (4.11) of [7]. When  $m > p-1$ , it follows from (4.12) of [7] and the assumption  $r_0 > r_{2(p-1)}$ .  $\square$

**Lemma 4.** *If  $r_0 = r_{2(p-1)}$ , then there exists a subsheaf  $W' \subset W$ , such that*

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \frac{r_{2(p-1)}(\mathrm{rk}(F_*W) - \mathrm{rk}(\mathcal{E}))}{p \cdot \mathrm{rk}(\mathcal{E}) \cdot \mathrm{rk}(W)} (\mu(W') - \mu(W/W'))$$

*Proof.* It is proved in [7] that there exists a subsheaf  $W' \subset W$  of rank  $r_{2(p-1)}$  such that  $\mathcal{F}_{2(p-1)} \cong W' \otimes T^{2(p-1)}(\Omega_X^1)$  and  $W' \otimes T^\ell(\Omega_X^1) \hookrightarrow \mathcal{F}_\ell$ . By (4.22) of [7], it is enough to show  $r'_\ell = 0$ , i.e.  $\mathrm{rk}(\mathcal{F}_\ell) = \mathrm{rk}(W' \otimes T^\ell(\Omega_X^1))$ , which follows from the local computations in the proof of Theorem 4.7 of [7].

For the convenience of readers, we repeat the arguments here. To show the assertion is a local problem. Let  $K = K(X)$  be the function field of  $X$  and consider the  $K$ -algebra

$$R = \frac{K[\alpha_1, \alpha_2]}{(\alpha_1^p, \alpha_2^p)} = \bigoplus_{\ell=0}^{2(p-1)} R^\ell,$$

where  $R^\ell$  is the  $K$ -linear space generated by

$$\{\alpha_1^{k_1} \alpha_2^{k_2} | k_1 + k_2 = \ell, 0 \leq k_i \leq p-1\}.$$

The quotients in the filtration (1) can be described locally

$$V_\ell/V_{\ell+1} = W \otimes_K R^\ell$$

as  $K$ -vector spaces. Then the homomorphism

$$\nabla : W \otimes_K R^\ell \rightarrow W \otimes_K R^{\ell-1} \otimes_K \Omega_{K/k}^1$$

in Theorem 2 is locally the  $k$ -linear homomorphism defined by

$$\nabla(w \otimes \alpha_1^{k_1} \alpha_2^{k_2}) = -w \otimes k_1 \alpha_1^{k_1-1} \alpha_2^{k_2} \otimes_K dx_1 - w \otimes k_2 \alpha_1^{k_1} \alpha_2^{k_2-1} \otimes_K dx_2.$$

And the fact that  $\mathcal{F}_\ell \xrightarrow{\nabla} \mathcal{F}_{\ell-1} \otimes \Omega_X^1$  for  $\mathcal{F}_\ell \subset W \otimes R^\ell$  is equivalent to

$$\forall \sum_j w_j \otimes f_j \in \mathcal{F}_\ell \Rightarrow \sum_j w_j \otimes \frac{\partial f_j}{\partial \alpha_i} \in \mathcal{F}_{\ell-1} (1 \leq i \leq 2). \quad (5)$$

The polynomial ring  $P = K[\partial_{\alpha_1}, \partial_{\alpha_2}]$  acts on  $R$  through partial derivations, which induces a D-module structure on  $R$ , where

$$D = \frac{K[\partial_{\alpha_1}, \partial_{\alpha_2}]}{(\partial_{\alpha_1}^p, \partial_{\alpha_2}^p)} = \bigoplus_{\ell=0}^{2(p-1)} D_\ell$$

and  $D_\ell$  is the linear space of degree  $\ell$  homogeneous elements. In particular,  $W \otimes R$  has the induced D-module structure with  $D$  acts on  $W$  trivially. Using this notation, (5) is equivalent to  $D_1 \cdot \mathcal{F}_\ell \subseteq \mathcal{F}_{\ell-1}$ .

Locally,  $\mathcal{F}_{2(p-1)}$  is equal to  $W' \otimes R^{2(p-1)}$  as  $K$ -vector spaces. Combining with  $D_1 \cdot \mathcal{F}_\ell \subseteq \mathcal{F}_{\ell-1}$ , we have

$$D_\ell \cdot \mathcal{F}_{2(p-1)} = W' \otimes D_\ell \cdot R^{2(p-1)} = W' \otimes R^{2(p-1)-\ell} \subset \mathcal{F}_{2(p-1)-\ell} \quad (6)$$

for  $0 \leq \ell \leq 2(p-1)$ , and the following sequence

$$W' = D_{2(p-1)} \cdot \mathcal{F}_{2(p-1)} \subseteq D_{2(p-1)-1} \cdot \mathcal{F}_{2(p-1)-1} \subseteq \cdots \subseteq D_1 \cdot \mathcal{F}_1 \subseteq \mathcal{F}_0.$$

But  $r_0 = r_{2(p-1)}$ , so  $\mathcal{F}_0 = W'$  and  $D_\ell \cdot \mathcal{F}_\ell = \mathcal{F}_0$  for  $1 \leq \ell \leq 2(p-1)$ . For any element  $\alpha \in \mathcal{F}_\ell \subset W \otimes R^\ell$ , it can be written as

$$\alpha = \sum w_{i_1 i_2} \otimes (\alpha_1^{i_1} \alpha_2^{i_2}),$$

where  $w_{i_1 i_2} \in W$  and the sum runs over  $i_1 + i_2 = \ell, 0 \leq i_1, i_2 \leq p-1$ . Meanwhile, we have

$$\partial_{\alpha_1}^{i_1} \partial_{\alpha_2}^{i_2} \cdot \sum w_{i_1 i_2} \otimes (\alpha_1^{i_1} \alpha_2^{i_2}) = w_{i_1 i_2} \in \mathcal{F}_0 = W'$$

from  $D_\ell \cdot \mathcal{F}_\ell = \mathcal{F}_0$ . Consequently,  $\alpha \in W' \otimes R^\ell$ , which implies that

$$\mathcal{F}_\ell \subseteq W' \otimes R^\ell.$$

Together with the conclusion  $W' \otimes R^\ell \subseteq \mathcal{F}_\ell$  in (6), we have

$$\mathcal{F}_\ell = W' \otimes R^\ell$$

for  $0 \leq \ell \leq 2(p-1)$ . Thus  $\text{rk}(\mathcal{F}_\ell) = \text{rk}(W' \otimes T^\ell(\Omega_X^1))$  for  $(0 \leq \ell \leq 2(p-1))$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

For any torsion free sheaf  $\mathcal{E}$ , we denote

$$s(\mathcal{E}) = \max_{\mathcal{F}} \{ \text{rk}(\mathcal{F})(\mu(\mathcal{F}) - \mu(\mathcal{E})) \mid \mathcal{F} \subseteq \mathcal{E} \}.$$

Then it is easy to see that  $s(\mathcal{E}) \geq 0$  and  $\mathcal{E}$  is semistable if and only if  $s(\mathcal{E}) = 0$ .

In this section, we always assume that  $X$  is a surface with  $\Omega_X^1$  semistable and  $\mu(\Omega_X^1) > 0$ ,  $W$  is a semistable bundle on  $X$  with  $\text{rk}(W) = r$ . In order to simplify the symbols, we denote  $A_\ell = \text{Sym}^\ell(\Omega_X^1) \otimes W$  and  $s(\ell) = s(A_\ell)$  for all  $\ell$ . Then we have the following lemmas.

**Lemma 5.** *As the above notations, we have*

$$s(\ell) - s(\ell-1) \leq s(\ell+1) - s(\ell).$$

*Proof.* Consider the exact sequence

$$0 \rightarrow A_{\ell-1} \otimes \omega_X \rightarrow A_\ell \otimes \Omega_X^1 \rightarrow A_{\ell+1} \rightarrow 0$$

where all of the bundles have the same slope  $(\ell + 1) \cdot \mu(\Omega_X^1) + \mu(W)$ . Assume  $\mathcal{E}_\ell$  is the subsheaf of  $A_\ell$  such that

$$\mathrm{rk}(\mathcal{E}_\ell) \cdot (\mu(\mathcal{E}_\ell) - \mu(A_\ell)) = s(\ell).$$

Then the above exact sequence induces an exact sequence

$$0 \rightarrow \mathcal{E}'_\ell \rightarrow \mathcal{E}_\ell \otimes \Omega_X^1 \rightarrow \mathcal{E}''_\ell \rightarrow 0,$$

where

$$\mathcal{E}'_\ell \subset A_{\ell-1} \otimes \omega_X, \quad \mathcal{E}''_\ell \subset A_{\ell+1}.$$

A direct computation implies

$$\begin{aligned} \mathrm{rk}(\mathcal{E}_\ell \otimes \Omega_X^1)(\mu(\mathcal{E}_\ell \otimes \Omega_X^1) - \mu(A_\ell \otimes \Omega_X^1)) \\ = \mathrm{rk}(\mathcal{E}'_\ell)(\mu(\mathcal{E}'_\ell) - \mu(A_{\ell-1} \otimes \omega_X)) + \mathrm{rk}(\mathcal{E}''_\ell)(\mu(\mathcal{E}''_\ell) - \mu(A_{\ell+1})) \end{aligned}$$

Consequently, we have

$$2s(\ell) \leq s(\ell - 1) + s(\ell + 1)$$

by the definition of  $s(\ell)$ . Thus

$$s(\ell) - s(\ell - 1) \leq s(\ell + 1) - s(\ell).$$

□

Taking  $\ell = p$  in (ii) of Proposition 3.5 of [6], we have the following exact sequence

$$0 \rightarrow W \otimes F^*\Omega_X^1 \rightarrow A_p \rightarrow A_{p-2} \otimes \omega_X \rightarrow 0,$$

we obtain a upper bound for  $s(\ell)$  by using the above exact sequence. For simplicity, we define  $t = s(W \otimes F^*\Omega_X^1)$ .

**Lemma 6.** *Assume  $p \geq r$ . Then we have*

$$s(\ell) \leq \frac{t}{2} \cdot (\ell - (p - r))$$

for  $p - r \leq \ell \leq p - 1$ .

*Proof.* Consider the exact sequence

$$0 \rightarrow W \otimes F^*\Omega_X^1 \rightarrow A_p \rightarrow A_{p-2} \otimes \omega_X \rightarrow 0$$

where all the bundles have the same slope  $p \cdot \mu(\Omega_X^1) + \mu(W)$ . As the same argument in Lemma 5, we have  $s(p) \leq t + s(p - 2)$ . Combining with  $2s(p - 1) \leq s(p) + s(p - 2)$ , we have

$$s(p - 1) - s(p - 2) \leq \frac{s(p) + s(p - 2)}{2} - s(p - 2) \leq \frac{t}{2}.$$

Then Lemma 5 implies that

$$s(\ell) - s(\ell - 1) \leq s(\ell + 1) - s(\ell) \leq \cdots \leq s(p - 1) - s(p - 2) \leq \frac{t}{2}.$$

But  $\text{Sym}^\ell(\Omega_X^1)$  is semistable for  $\ell \leq p-1$  and

$$\text{rk}(\text{Sym}^\ell(\Omega_X^1)) + \text{rk}(W) = \ell + 1 + r \leq p + 1$$

for  $\ell \leq p-r$ , thus we have  $A_\ell$  is semistable for  $\ell \leq p-r$  by a theorem of Ilangoan-Mehta-Parameswaran (see Section 6 of [3] for the precise statement): If  $E_1, E_2$  are semistable with  $\text{rk}(E_1) + \text{rk}(E_2) \leq p+1$ , then  $E_1 \otimes E_2$  is semistable. Consequently, we have  $s(\ell) = 0$  for  $\ell \leq p-r$ . Then the result is a direct computation.  $\square$

**Lemma 7.** *Assume  $p \geq r+1$ . Then we have*

$$t \leq (2r-1) \cdot \mu(\Omega_X^1).$$

*Proof.* By the proposition 3.9 of [7], We have  $I(F^*\Omega_X^1) \leq \mu(\Omega_X^1)$ , If  $p \geq r+1$ , then it is easy to check that

$$I(W \otimes F^*\Omega_X^1) = I(F^*\Omega_X^1).$$

Thus we have

$$t \leq (2r-1) \cdot I(W \otimes F^*\Omega_X^1) \leq (2r-1) \cdot \mu(\Omega_X^1).$$

$\square$

Now, we finish the proof of Theorem 1.

*Proof of Theorem 1:* Let us assume that  $W$  is semistable firstly.

If  $r_0 = r_{2(p-1)}$ , then Lemma 4 implies that there exists a subsheaf  $W' \subset W$  such that

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \frac{r_{2(p-1)}(\text{rk}(F_*W) - \text{rk}(\mathcal{E}))}{p \cdot \text{rk}(\mathcal{E}) \cdot \text{rk}(W)} (\mu(W') - \mu(W/W')) \leq 0$$

If  $r_0 > r_{2(p-1)}$ , then we have

$$\sum_{\ell=0}^m (p-1-\ell)r_\ell \geq (p-1)$$

by Lemma 3. Consider formula (4), it is enough to prove that

$$\sum_{\ell=0}^m r_\ell (\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) \leq (p-1) \cdot \mu(\Omega_X^1).$$

Recall that  $V_\ell/V_{\ell+1} = W \otimes T^\ell(\Omega_X^1)$ , where

$$T^\ell(\Omega_X^1) = \begin{cases} \text{Sym}^\ell(\Omega_X^1) & \text{when } \ell < p \\ \text{Sym}^{2(p-1)-\ell}(\Omega_X^1) \otimes \omega_X^{\ell-(p-1)} & \text{when } \ell \geq p. \end{cases}$$



Consequently, we have  $V_\ell/V_{\ell+1}$  is semistable for  $\ell \leq p - r$  and  $\ell \geq p + r - 2$ , and we only need to prove

$$\sum_{\ell=p-r+1}^{p+r-3} r_\ell(\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) \leq (p-1) \cdot \mu(\Omega_X^1).$$

But

$$r_\ell(\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) \leq s(2(p-1) - \ell)$$

for  $p \leq \ell \leq p + r - 3$ . Combining with Lemma 6 and Lemma 7, we obtain that

$$\begin{aligned} \sum_{\ell=p-r+1}^{p+r-3} r_\ell(\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) &= \sum_{\ell=p-r+1}^{p-1} r_\ell(\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) + \sum_{\ell=p}^{p+r-3} r_\ell(\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})) \\ &\leq \sum_{\ell=p-r+1}^{p-1} s(\ell) + \sum_{\ell=p}^{p+r-3} s(2(p-1) - \ell) \\ &\leq \frac{t}{2} \cdot (1 + \cdots + r-1) + \frac{t}{2} \cdot (r-2 + \cdots + 1) \\ &\leq \frac{1}{2}(2r-1)(r-1)^2 \cdot \mu(\Omega_X^1) \\ &\leq (p-1) \cdot \mu(\Omega_X^1) \end{aligned}$$

If  $W$  is stable, we can prove that  $F_*W$  is stable similarly. The proof is completed.  $\square$

**Remark 1.** *Keep the assumption of Theorem 1. For  $r = 1$ , the stability of  $F_*W$  is proved by X. Sun in [7]. As a slightly generalized version of [2, Theorem 3.1], it is proved by X. Sun that  $F_*(L \otimes \Omega_X^1)$  is semistable when  $L$  is a line bundle; moreover, if  $\Omega_X^1$  is stable, then  $F_*(L \otimes \Omega_X^1)$  is stable (see [7, Theorem 4.9]). There is no restriction on the characteristic  $p$  for these results.*

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#### REFERENCES

- [1] N.Katz, Nilpotent connection and the monodromy theorem: Application of a result of Turrittin, I.H.E.S. Publ. Math., **39** (1970), 175-232
- [2] Y.Kitadai and H.Sumihoro, Canonical filtrations and stability of direct images by Frobenius morphisms II, Hiroshima Math. J., **38** (2008), 243-261.

- [3] A. Langer, Semistable sheaves in positive characteristic, *Ann. of Math.* (2), **159** (2004), 251-276.
- [4] L. Li and F. Yu, Instability of truncated symmetric powers of sheaves, *J. Algebra*, **386** (2013), 176-189.
- [5] V. Mehta and C. Pauly, Semistability of Frobenius direct images over curves, *Bull. Soc. Math. France*, **135** (2007), 105-117.
- [6] X. Sun, Direct images of bundles under Frobenius morphism, *Invent. Math.*, **173** (2008), 427-447.
- [7] X. Sun, Frobenius morphism and semistable bundles, *Advanced Studies in Pure Mathematics* **60** (2010), Algebraic Geometry in East Asia-Seoul (2008), 161-182.

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